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Level crossing analysis of Burgers equation in 1 + 1 dimensions

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Abstract
We investigate the average frequency of positive slope ν_α^+, crossing the velocity field $u(x) - \bar{u} = \alpha$ in the Burgers equation. The level crossing analysis in the inviscid limit and the total number of positive crossings of the velocity field before the creation of singularities are given. The main goal of this paper is to show that this quantity, ν_α^+, is a good measure for the fluctuations of velocity fields in the Burgers turbulence.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

The Burgers equation is the simplest nonlinear generalization of the diffusion equation. The N-dimensional forced Burgers equation

$$\partial_t \vec{u} + (\vec{u}.\vec{\nabla})\vec{u} = \nu \nabla^2 \vec{u} + \vec{f}(\vec{x}, t), \tag{1}$$

which describes the dynamics of a stirred, pressureless and vorticity-free fluid, has found interesting applications in a wide range of non-equilibrium statistical physics problems. It arises, for instance, in cosmology where it is known as the adhesion model [1], in vehicular traffic [2] or in the study of directed polymers in random media [3]. In the Burgers equation if the velocity field is a gradient field $\vec{u}(\vec{x}, t) = -\vec{\nabla}\psi(\vec{x}, t)$ and the random force is a gradient random force $f(\vec{x}, t) = -\vec{\nabla}F(\vec{x}, t)$, then the associated Hamilton–Jacobi equation satisfies the following equation as

$$\partial_t \psi = \nu \nabla^2 \psi + \tfrac{1}{2}(\vec{\nabla}\psi)^2 + F(\vec{x}, t), \tag{2}$$

where ν is the viscosity; recently it has been frequently studied as a nonlinear model for the motion of an interface under deposition [4]. The case with large-scale forcing was considered in [5, 6] as a natural way to pump energy in order to maintain a statistical steady state. The Burgers equation is then a simple model for studying the influence of well-understood structures (shocks, preshocks, etc) on the statistical properties of the flow. As is well known, equation (1) in the limit of vanishing viscosity ($\nu \to 0$) displays after a finite time dissipative singularities, namely shocks, corresponding to discontinuities in the velocity field. In the presence of large-scale forcing, it was recently stressed for the one-dimensional case [7, 8] and also for higher dimensions [9–11] that the global topological structure of such singularities is strongly related to the boundary conditions associated with the equation. More precisely, when, for instance, space periodicity is assumed, a generic topological shock structure can be outlined. It plays an essential role in understanding the qualitative features of the statistically stationary regime. So far, the singular structure of the forced Burgers equation was mostly investigated in the case of finite-size systems with periodic boundary conditions. It is however frequently of physical interest to investigate instances where the size of the domain is much larger than the scale, so as to examine, for example, the role of Galilean invariance [12].

Here we describe the level crossing analysis in the context of vorticity-free fluid. In the level crossing analysis we are interested in determining the average frequency (in spatial dimension) of observing the definite value for the velocity function $u(x) - \bar{u} = \alpha$ in fluid, ν_α^+, from which one can find the average number of crossing the given velocity in a sample with size L. The average number of visiting the velocity $u(x) - \bar{u} = \alpha$ with positive slope will be $N_\alpha^+ = \nu_\alpha^+ L$. It can be shown that the ν_α^+ can be written sin terms of joint probability distribution function (PDF) of $u(x) - \bar{u}$ and its gradient. Therefore the quantity ν_α^+ carries the whole information of fluid which lies in joint PDF of velocity and its gradient fluctuations. This work aims to study the frequency of positive slope crossing (i.e. ν_α^+) in time t on the vorticity-free fluid in a sample with size L. We describe a quantity N_{tot}^+ which is defined as $N_{\text{tot}}^+ = \int_{-\infty}^{+\infty} \nu_\alpha^+ \, \mathrm{d}\alpha$ to measure the total number of crossing the velocity of fluid with positive slope. The N_{tot}^+ and the path which is constructed by the velocity of fluid are in the same order. It is expected that in the stationary state the N_{tot}^+ to become size dependent. Although we exactly determine the velocity dependence of ν_α^+ for the Burgers equation in the inviscid limit and before creation of singularities, we compute the time dependence of $N_{\text{tot}}^+ \left(\nu_\alpha^+\right)$ numerically.

This paper is organized as follows: in section 2 we discuss the connection between ν_α^+ and the underlying probability distribution functions (PDF) of a fluid [13]. In section 3 we derive the integral representation of ν_α^+ for the Burgers equation in 1+1 dimensions and in the inviscid limit before the creation of singularities. Section 4 closes with a discussion of the present results.

2. The level crossing analysis of stochastic processes

Consider a sample function of an ensemble of functions which make up the homogeneous random process $u(x, t)$. Let n_α^+ denote the number of positive slope crossing of $u(x) - \bar{u} = \alpha$ in time t for a typical sample size L (see figure 1) and let the mean value for all the samples be $N_\alpha^+(L)$ where

$$N_\alpha^+(L) = E\left[n_\alpha^+(L)\right]. \tag{3}$$

Since the process is homogeneous, if we take a second interval of L immediately following the first we shall obtain the same result, and for the two intervals together we shall therefore obtain

$$N_\alpha^+(2L) = 2N_\alpha^+(L), \tag{4}$$

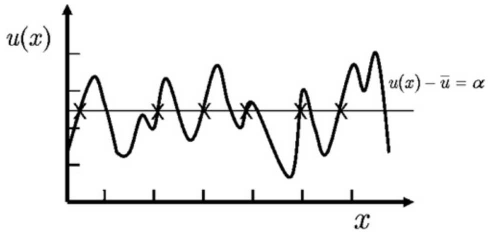

Figure 1. Positive slope crossing of the level $u(x) - \bar{u} = \alpha$.

from which it follows that, for a homogeneous process, the average number of crossing is proportional to the space interval L. Hence

$$N_\alpha^+(L) \propto L, \tag{5}$$

or

$$N_\alpha^+(L) = \nu_\alpha^+ L, \tag{6}$$

where ν_α^+ is the average frequency of positive slope crossing of the level $u(x) - \bar{u} = \alpha$. We now consider how the frequency parameter ν_α^+ can be deduced from the underlying probability distributions for $u(x) - \bar{u}$. Consider a small length dl of a typical sample function. Since we are assuming that the process $u(x) - \bar{u}$ is a smooth function of x, with no sudden ups and downs, if dl is small enough, the sample can only cross $u(x) - \bar{u} = \alpha$ with positive slope if $u(x) - \bar{u} < \alpha$ at the beginning of the interval location x. Furthermore there is a minimum slope at position x if the level $u(x) - \bar{u} = \alpha$ is to be crossed in the interval dl depending on the value of $u(x) - \bar{u}$ at the location x. So there will be a positive crossing of $u(x) - \bar{u} = \alpha$ in the next space interval dl if, at position x,

$$u(x) - \bar{u} < \alpha \qquad \text{and} \qquad \frac{\mathrm{d}(u(x) - \bar{u})}{\mathrm{d}l} > \frac{\alpha - (u(x) - \bar{u})}{\mathrm{d}l}. \tag{7}$$

Actually what we really mean is that there will be a high probability of crossing in the interval dl if these conditions are satisfied [14, 15].

In order to determine whether the above conditions are satisfied at any arbitrary location x, we must find how the values of $y = u(x) - \bar{u}$ and $y' = \frac{\mathrm{d}y}{\mathrm{d}l}$ are distributed by considering their joint probability density $p(y, y')$. Suppose that the level $y = \alpha$ and interval dl are specified. Then we are only interested in values of $y < \alpha$ and values of $y' = \left(\frac{\mathrm{d}y}{\mathrm{d}l}\right) > \frac{\alpha - y}{\mathrm{d}l}$, which means that the region between the lines $y = \alpha$ and $y' = \frac{\alpha - y}{\mathrm{d}l}$ in the plane (y, y'). Hence the probability of positive slope crossing of $y = \alpha$ in dl is

$$\int_0^\infty \mathrm{d}y' \int_{\alpha - y'\,\mathrm{d}l}^{\alpha} \mathrm{d}y \; p(y, y'). \tag{8}$$

When d$l \to 0$, it is legitimate to put

$$p(y, y') = p(y = \alpha, y'). \tag{9}$$

Since at large values of y and y' the probability density function approaches zero fast enough; therefore equation (8) may be written as

$$\int_0^\infty \mathrm{d}y' \int_{\alpha - y'\,\mathrm{d}l}^{\alpha} \mathrm{d}y \; p(y = \alpha, y'), \tag{10}$$

in which the integrand is no longer a function of y so that the first integral is just $\int_{\alpha-y'\,\mathrm{d}l}^{\alpha} \mathrm{d}y\, p(y=\alpha, y') = p(y=\alpha, y')y'\,\mathrm{d}l$, so that the probability of slope crossing of $y=\alpha$ in $\mathrm{d}l$ is equal to

$$\mathrm{d}l \int_0^\infty p(\alpha, y')y'\,\mathrm{d}y', \tag{11}$$

in which the term $p(\alpha, y')$ is the joint probability density $p(y, y')$ evaluated at $y=\alpha$.

We have said that the average number of positive slope crossing in the scale L is $\nu_\alpha^+ L$, according to equation (6). The average number of crossing in the interval $\mathrm{d}l$ is therefore $\nu_\alpha^+\,\mathrm{d}l$. So the average number of positive crossing of $y=\alpha$ in interval $\mathrm{d}l$ is equal to the probability of positive crossing of $y=a$ in $\mathrm{d}l$, which is only true because $\mathrm{d}l$ is small and the process $y(x)$ is smooth so that there cannot be more than one crossing of $y=\alpha$ in the space interval $\mathrm{d}l$. Therefore we have $\nu_\alpha^+\,\mathrm{d}l = \mathrm{d}l\int_0^\infty p(\alpha, y')y'\,\mathrm{d}y'$, from which we get the following result for the frequency parameter ν_α^+ in terms of the joint probability density function $p(y, y')$ as follows:

$$\nu_\alpha^+ = \int_0^\infty p(\alpha, y')y'\,\mathrm{d}y'. \tag{12}$$

In the following section we are going to derive ν_α^+ via the joint PDF of $u(x)-\bar{u}$ and the velocity gradient. To derive the joint PDF we use the master equation method [16]. This method enables us to find ν_α^+ in terms of the generating function.

3. Frequency of a definite velocity with positive slope for the Burgers equation before the singularity formation

As mentioned in section 1, in the presence of a random force $\vec{f}(\vec{x}, t)$, the velocity field of burgers equation in 1 + 1 dimensions evolves as

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2} + f(x,t). \tag{13}$$

Differentiating the Burgers equation (equation (13)) with respect to x, we have

$$\frac{\partial \omega}{\partial t} + u\frac{\partial \omega}{\partial x} + \omega^2 = \nu\frac{\partial^2 \omega}{\partial x^2} + f_x(x,t), \tag{14}$$

where $\nu \geqslant 0$, $\omega = \frac{\partial u}{\partial x}$ and $f(x,t)$ is a random force, with a Gaussian distribution of mean zero and second moment given by

$$\langle f(x,t)f(x',t')\rangle = 2D_0 D(x-x')\delta(t-t'), \tag{15}$$

where $D(x)$ is the space correlation function and is an even function of its argument. It has the following form:

$$D(x-x') = \frac{1}{\sqrt{\pi}\sigma}\,\mathrm{e}^{-\frac{(x-x')^2}{\sigma^2}}, \tag{16}$$

where σ is the standard deviation of $D(x-x')$. Typically in the realistic problem, the correlation of forcing is considered as a smooth function for mimicking the long-range correlation. We regularize the smooth function correlation by a Gaussian function. When the variance σ is in order of the system size, we would expect that the model would represent a long-range character for the forcing. So we should stress that our calculations are done for finite $\sigma \sim L$, where L is the system size. Parameters ν and D_0 are describing the kinematics viscosity and the noise strength, respectively. Once trying to develop the statistical theory of the Burgers equation it becomes clear that the inter-dependency of the velocity field and velocity gradient statistics would be taken into account. The very existence of the nonlinear term in the Burgers equation leads to development of the singularities in a *finite time* and in the inviscid limit, i.e. $\nu \to 0$. So one would distinguish between different time regimes. Recently it has been shown that starting from the flat interface the Burgers equation will develop shock singularity after a time scale t^*, where t^* depends on the forcing properties as $t^* \simeq {D_0}^{-1/3}\sigma$ [11, 16, 17]. This means that for time scales before t^* the relaxation contribution tends to zero when $\nu \to 0$. In this regime one can observe that the generating function equation is closed [11, 17–19]. Let us define the generating function $Z(\lambda, \mu, x, t)$ as

$$Z(\lambda, \mu, t) = \langle \mathrm{e}^{-\mathrm{i}\lambda(u(x,t)-\bar{u})-\mathrm{i}\mu\omega(x,t)} \rangle = \langle \Theta \rangle. \tag{17}$$

Assuming statistical homogeneity i.e. $Z_x = 0$ it follows from equations (13) and (14) that Z satisfies in the following equation:

$$\frac{\partial Z}{\partial t} = -\mathrm{i}\lambda \left\langle \frac{\partial u}{\partial t} \Theta \right\rangle - \mathrm{i}\mu \left\langle \frac{\partial^2 u}{\partial t \partial x} \Theta \right\rangle. \tag{18}$$

Using the Novikove theorem [11] gives

$$\frac{\partial Z}{\partial t} = \mathrm{i}Z_\mu - \mathrm{i}\mu Z_{\mu,\mu} - \lambda^2 k(0) Z + \mu^2 k_{xx}(0) Z, \tag{19}$$

where $k(x - x') = 2D_0 D(x - x'), k(0) = \frac{2D_0}{\sqrt{\pi}\sigma}$ and $k_{xx}(0) = -\frac{4D_0}{\sqrt{\pi}\sigma^3}$. The solution of equation (19) by using the separation of variables is as follows:

$$Z(t, \mu, \lambda) = \sqrt{\frac{k(0)t}{\pi}}\, \mathrm{e}^{-\lambda^2 K(0)t} \times Z_1(\mu, t), \tag{20}$$

where $Z_1(\mu, t)$ satisfies the following equation:

$$\frac{\partial Z_1}{\partial t} = \mathrm{i}Z_{1\mu} - \mathrm{i}\mu Z_{1\mu,\mu} - \lambda^2 k(0) Z_1 + [\mu^2 k_{xx}(0) + C(t)] Z_1, \tag{21}$$

$C(t)$ is an arbitrary function which should be determined by the initial conditions.

The joint probability density function of u and ω can be obtained by the Fourier transform of the generating function:

$$P(u, \omega, t) = \frac{1}{2\pi} \int \mathrm{d}\lambda\, \mathrm{d}\mu\, \mathrm{e}^{\mathrm{i}\lambda(u-\bar{u})+\mathrm{i}\mu\omega} Z(\lambda, \mu, t), \tag{22}$$

so by Fourier transforming equation (19) we get the Fokker–Planck equation as

$$\frac{\partial}{\partial t} P = 3\omega P + \omega^2 \frac{\partial P}{\partial \omega} + k(0) \frac{\partial^2 P}{\partial u^2} - k_{xx}(0) \frac{\partial^2 P}{\partial \omega^2}. \tag{23}$$

The solution of the above equation can be separated as $P(u, \omega, t) = p_1(u, t) p_2(\omega, t)$ (for motivation see [20]). Using the initial conditions $P_1(u, 0) = \delta(u)$ and $P_2(\omega, 0) = \delta(\omega)$ it can be shown that

$$P(u, \omega, t) = \frac{1}{\sqrt{4\pi k(0)t}}\, \mathrm{e}^{-\frac{u^2}{4K(0)t}} \times p_2(\omega, t), \tag{24}$$

where $p_2(\omega, t)$ is a solution of the following equation:

$$\frac{\partial}{\partial t} p_2 = +\omega^2 \frac{\partial p_2}{\partial \omega} - k_{xx}(0) \frac{\partial^2 p_2}{\partial \omega^2} + [3\omega + G(t)] p_2(\omega, t), \tag{25}$$

$G(t)$ is an arbitrary function which should be determined by the initial conditions. Up to now we obtained that, for every time scale, the level cross ν_α^+ has a Gaussian behaviour in terms of α, now for determination of time dependence of ν_α^+ we have to determine $p_2(\omega, t)$. Since equation (25) is so complex, we solve it using the numerical methods [21]. The frequency

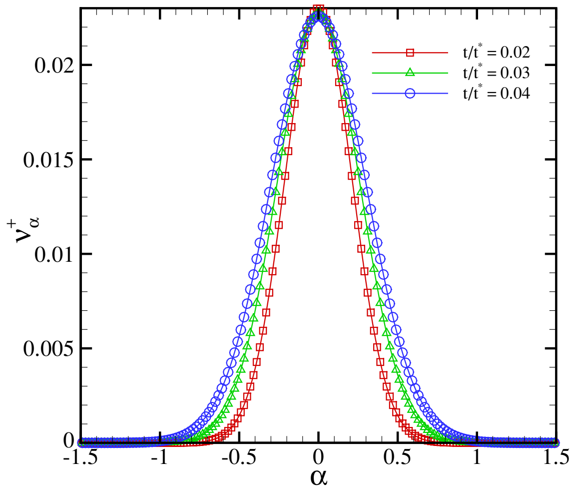

Figure 2. Plot of ν_α^+ versus α for the Burgers equation in the strong coupling and before the creation of singularity for time scale $t/t* = 0.02$, 0.03 and 0.04.

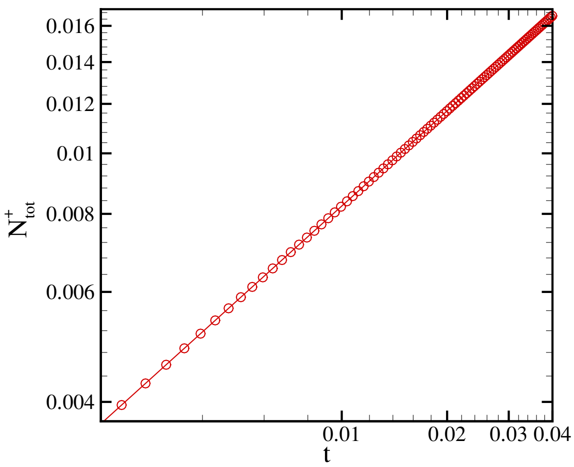

Figure 3. Log–log plot of $N_{\rm tot}^+$ versus t for the Burgers equation in the strong coupling before the creation of singularity in the velocity field.

of repeating a definite velocity field $(u(x) - \bar{u} = \alpha)$ with positive slope can be calculated as $\nu_\alpha^+ = \int_0^\infty \omega P(\alpha, \omega, t)\, d\omega$. Figure 2 shows ν_α^+ for various time scales before creation of singularity. To derive the $N_{\rm tot}^+$ let us express $N_{\rm tot}^+$ as

$$N_{\rm tot}^+ = \int_{-\infty}^{+\infty} d\alpha \int_0^\infty \omega P(\alpha, \omega, t)\, d\omega. \tag{26}$$

Using the numerical integration of equation (26) one finds $N_{\rm tot}^+ \sim t^\beta$ where $\beta = 0.50 \pm 0.01$, In figure 3 we plot $N_{\rm tot}^+$ as a function of t.

4. Conclusion

We obtained some results in the problems of Burgers equation in 1 + 1 dimensions with a Gaussian forcing which is white in time and Gaussian correlated in space, typically in the physical case. We determined the average frequency of crossing, i.e. ν_α^+ of observing the definite value for the velocity field $u - \bar{u} = \alpha$, from which one can find the average number of crossing the given velocity field in a sample with size L. The integral representation of ν_α^+ was given for the Burgers equation in the inviscid limit before the creation of singularity and it was shown that the velocity dependence of the ν_α^+ is Gaussian. We apply the quantity $N_{\text{tot}}^+ = \int_{-\infty}^{+\infty} \nu_\alpha^+ \, \mathrm{d}\alpha$, which measures the total number of positive crossing of velocity and show that for the Burgers equation in the inviscid limit and before the creation of singularities N_{tot}^+ scales as $t^{1/2}$.

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